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Magnetic permeability of the Dirac vacuum under confining boundary conditions

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Abstract

We obtain the magnetic properties of the Dirac vacuum confined between two parallel plates. The confinement is implemented by MIT boundary conditions on the Dirac field and the properties are described by corrections to the Euler–Heisenberg effective Lagrangian. When the plates are separated by a distance in the range from nanometres to micrometres the new term associates with the vacuum a magnetic susceptibility of the same order of that of hydrogen or nitrogen at room temperature and atmospheric pressure.

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1. Introduction

Classical electromagnetic fields in classical vacuum are described by the Maxwell Lagrangian density $\mathcal{L}^{(0)} = (\mathbf{E}^2 - \mathbf{B}^2)/2$. The same fields in a medium are effectively described by a Lagrangian density $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}$, where $\mathcal{L}^{(1)}$ is a function of the fields $\mathbf{E}$ and $\mathbf{B}$ which takes into account the net result of the interaction between the medium and the electromagnetic field. The Lagrangian $\mathcal{L}$, or rather the contribution $\mathcal{L}^{(1)}$ from the medium, is called effective Lagrangian density of the electromagnetic field in the medium. Particularly important are the terms in $\mathcal{L}^{(1)}$ which are quadratic in the fields $\mathbf{E}$ and $\mathbf{B}$. They can be made explicit by writing $\mathcal{L}^{(1)} = [(\varepsilon - 1)\mathbf{E}^2 - (\mu^{-1} - 1))\mathbf{B}^2]/2 + \mathcal{L}'^{(1)}$, where ε and μ are the electric and magnetic permeability constants (or tensors, if anisotropy is present) of the medium and $\mathcal{L}'^{(1)}$ contains the higher order terms in the electromagnetic fields. The quadratic terms can be added to the Maxwell Lagrangian density and we end up with the complete Lagrangian density in the form

$$\mathcal{L} = \frac{\varepsilon}{2}\mathbf{E}^2 - \frac{1}{2\mu}\mathbf{B}^2 + \mathcal{L}'^{(1)}. \tag{1}$$

From ε and μ we get the linear constitutive properties of the medium, while $\mathcal{L}'^{(1)}$ is responsible for the nonlinear properties. Some distinguished properties of a medium are the speed of light propagating in it, and its birefringence and dichroism in the case of anisotropy.

Vacuum in quantum field theory, with its perpetual quantum fluctuations, is very far from the complete emptiness of the classical vacuum and presents several properties of a material medium as, e.g., polarizability under the influence of an external electromagnetic field. Under such an influence, the one-loop constitutive properties of the QED vacuum are described by the Euler–Heisenberg effective Lagrangian density [1–3]—provided the external field is a slowly varying function of space and time (obeying the soft photon approximation, $\hbar\omega \ll mc$, where ω is the typical field frequency and m is the electron mass). These constitutive properties turn out to be nonlinear functions of the applied electromagnetic field, which may provide anisotropy for birefringence in vacuum. These properties are observable, in principle, if the applied field approaches the critical value m^2/e, where e is the electron charge. Such critical value, around 10^{18} V m^{-1} for the electric field and 10^9 T for the magnetic field, is still too high for experimental feasibility. However, with the much less intense field of 5 T rotation of the polarization plane of light has been recently observed by Zavattini *et al* [4] (see also [5–7]). The nonlinearity of constitutive properties means that in (1) the constitutive constants ε and μ are equal to 1 and the only correction to Maxwell's Lagrangian comes from $\mathcal{L}'^{(1)}$. However, if boundary conditions are imposed to the Dirac vacuum, they not only modify the nonlinear effects described by $\mathcal{L}'^{(1)}$ but also give rise to new effects of a linear character. It has been shown that antiperiodic boundary conditions on the Dirac field along a spatial direction endows the vacuum with a greater than 1 permeability obtained from one-loop contributions [8], but with negligible value in scales accessible to direct observation, say around micrometres or nanometres.

Although anti-periodic boundary conditions are suited for analysing compactification of extra dimensions or as a theoretical laboratory for the investigation of new phenomena, the situation of immediate physical interest is the confinement of the matter field in a region of space as, e.g., in the case of quarks inside a hadron. This confinement is implemented by boundaries which are impermeable to the Dirac current, as described by the MIT boundary conditions [9]. In this way we are faced with the much more difficult problem of calculating the Euler–Heisenberg effective Lagrangian density when the Dirac field is confined by MIT boundary conditions. Here, we present the solution for this problem obtaining new contributions to the Euler–Heisenberg effective Lagrangian. As a consequence of considering the more realistic situation described by the MIT boundary conditions, we will be rewarded with a significant increase in the order of magnitude for the resulting vacuum magnetic permeability from its value in the case of antiperiodic boundary conditions, as we will see in what follows.

We are here interested in the contribution to the vacuum magnetic properties from the Dirac field confined by MIT boundary conditions, but the joint external influence of magnetic field and such boundary conditions on the vacuum can also be used to investigate the Dirac Casimir effect [10] under the influence of an external magnetic field as was done by Elizalde, Santos and Tort [11] (for an introduction to Casimir effect see [12] and for recent reviews [13, 14]). It is found that the usual Casimir effect for the Dirac field with MIT boundary conditions (without the influence of external fields) [15–17] is significantly modified by the introduction of the external magnetic field [11].

The constitutive properties of QED vacuum were already considered by Bordag [18] in an investigation of the radiative corrections to photon state between superconducting plates (see also [19]). He found a shift in photon energy which can be interpreted as a renormalization of the distance between the plates. The photon experiences an enlarged distance between them and so a smaller speed of light. The calculation of the electric permittivity and the magnetic permeability of QED vacuum between perfectly conducting plates was performed by Scharnhorst [20]. He used these quantities to obtain the change in the index of refraction between the plates. This gives rise to a change in the speed of light which propagates between and perpendicular to the plates, the so-called Scharnhorst effect. By its significance it is an important effect as a matter of first principles [21], although too small to be directly observable in present day experiments. The Scharnhorst effect is a two-loop effect—even though it can also be obtained from the (one-loop) Euler–Heisenberg Lagrangian density by an ingenious method proposed by Barton [21]. In those previous works the confining boundary conditions are imposed only on the electromagnetic field, while here we consider the MIT boundary conditions on the matter field and no boundary conditions on the electromagnetic field. As a consequence, the resulting effects appear already at the one-loop level.

2. Effective Lagrangian and permeability

Let us consider the Dirac field with mass m and charge e and the confining parallel plates as squares of side ℓ much larger than the separation a and perpendicular to the z-axis, say at $z = 0$ and $z = a$. Since the plates are impermeable to Dirac currents we may call them dielectric plates. A magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is applied perpendicularly to them and we shall assume, without loss of generality, that eB is positive. The confinement of the Dirac field Ψ between the plates is implemented by the following MIT boundary conditions:

$$(\mathbb{1} - \mathrm{i}\gamma^3)\Psi|_{z=0} = 0 \qquad \text{and} \qquad (\mathbb{1} + \mathrm{i}\gamma^3)\Psi|_{z=a} = 0, \tag{2}$$

where γ^3 is the Dirac matrix associated with the z-direction. The frequency spectrum of the field is given by

$$\omega_{nl} = \sqrt{p_l^2 + 2eBn + m^2}, \tag{3}$$

where n are the non-negative integers labelling the Landau levels and p_l are the positive solutions to the equation $f(pa) = 0$, with the function f determined by the MIT boundary condition (2) as

$$f(pa) = am\sin(pa) + pa\cos(pa). \tag{4}$$

Equation (4) also has zero and negative solutions, but they do not correspond to linearly independent eigenvectors. For the Landau levels with positive n there is a double degeneracy which does not occur for the $n = 0$ level, and for all levels there is a well-known degeneracy $eB\ell^2/2\pi$ stemming from the transverse degrees of freedom. With this energy spectrum at hand we use the Weisskopf original method [2] for calculating the effective Lagrangian density $\mathcal{L}^{(1)}$ from the sum over modes,

$$\mathcal{L}^{(1)}(B,a)a\ell^2 = 2\frac{eB\ell^2}{2\pi}\sum_{n=(0)}^{\infty}\sum_{l=1}^{\infty}\omega_{nl}\,\mathrm{e}^{-\lambda\omega_{nl}/m}, \tag{5}$$

where $n = (0)$ indicates that the $n = 0$ term in the sum must be multiplied by $1/2$ in order to compensate the factor 2 multiplying the sum to account for the double degeneracy of the other levels, and the exponential cut-off depending on the positive parameter λ is used to control ultraviolet divergences. To sum over the positive zeros of the function f defined in (4) we use Cauchy's integral [22],

$$\sum_{l=1}^{\infty}\omega_{nl}\,\mathrm{e}^{-\lambda\omega_{nl}/m} = \frac{1}{2\pi\mathrm{i}}\oint_{C}\mathrm{d}z\sqrt{z^2+2eBn+m^2}\,\mathrm{e}^{-\lambda\sqrt{z^2+2eBn+m^2}/m}\frac{\mathrm{d}}{\mathrm{d}z}\log f(za), \tag{6}$$

where we should consider the limit in which the curve C has all the positive zeros in its interior.

In this way, we obtain

$$\sum_{l=1}^{\infty} \omega_{nl}\, e^{-\lambda\omega_{nl}/m} = -\frac{1}{2}\sqrt{2bn+1}\, e^{-\lambda\sqrt{2bn+1}}$$
$$-\frac{m}{\pi}\int_{\sqrt{2bn+1}}^{\infty} dy\sqrt{y^2-2bn-1}\cos\left(\lambda\sqrt{y^2-2bn-1}\right)$$
$$\times \frac{d}{dy}\log[\sinh(amy)+y\cosh(amy)], \tag{7}$$

where $b = eB/m^2$ is the magnetic field in units of the critical field $B_{cr} = m^2/e$. Using the identity

$$\frac{d}{dy}\log[\sinh(amy)+y\cosh(amy)] = am + \frac{1}{y+1} + \frac{d}{dy}\log\left[1+\frac{y-1}{y+1}\, e^{-2amy}\right] \tag{8}$$

and changing the integration variable to $x = \sqrt{y^2-2bn-1}$, we obtain the following expression for the effective Lagrangian density:

$$\mathcal{L}^{(1)}(B,a) = \mathcal{L}^{(1)}_{EH}(B) + \tilde{\mathcal{L}}^{(1)}(B,a), \tag{9}$$

where

$$\mathcal{L}^{(1)}_{EH}(B) = \frac{m^4 b}{\pi^2}\frac{\partial^2}{\partial\lambda^2}\sum_{n=(0)}^{\infty}\int_0^{\infty} dx\,\frac{\cos\lambda x}{\sqrt{x^2+2bn+1}} \tag{10}$$

and

$$\tilde{\mathcal{L}}^{(1)}(B,a) = \frac{m^3 b}{2\pi a}\frac{\partial}{\partial\lambda}\sum_{n=(0)}^{\infty} e^{-\lambda\sqrt{2bn+1}}$$
$$+\frac{m^3 b}{\pi^2 a}\frac{\partial^2}{\partial\lambda^2}\sum_{n=(0)}^{\infty}\int_0^{\infty} dx\,\frac{\cos\lambda x}{x^2+2bn+1+\sqrt{x^2+2bn+1}}$$
$$+\frac{m^3 b}{\pi^2 a}\sum_{n=(0)}^{\infty}\int_0^{\infty} dx\,\frac{d}{dx}(x\cos\lambda x)$$
$$\times\log\left(1+\frac{\sqrt{x^2+2bn+1}-1}{\sqrt{x^2+2bn+1}+1}\, e^{-2am\sqrt{x^2+2bn+1}}\right). \tag{11}$$

Using the identity (formula (3.381) in [23])

$$\frac{1}{\sqrt{x^2+2bn+1}} = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{ds}{s^{1/2}}\, e^{-s(x^2+2bn+1)}, \tag{12}$$

$\mathcal{L}^{(1)}_{EH}$ can be written as

$$\mathcal{L}^{(1)}_{EH}(B) = -\frac{m^4 b}{8\pi^2}\int_0^{\infty}\frac{ds}{s^2}\left(1-\frac{\lambda^2}{2s}\right) e^{-s-\lambda^2/4s}\coth(bs), \tag{13}$$

and using the identities (formula (3.472.5) in [23])

$$e^{-\lambda\sqrt{2bn+1}} = \frac{\lambda}{2\sqrt{\pi}}\int_0^{\infty}\frac{ds}{s^{3/2}}\, e^{-s(2bn+1)-\lambda^2/4s} \tag{14}$$

and (formula (6.283.1) in [23])

$$\frac{1}{x^2+2bn+1+\sqrt{x^2+2bn+1}} = \int_0^\infty \mathrm{d}s\, \mathrm{erfc}(\sqrt{s})\, \mathrm{e}^{-s(x^2+2bn)}, \tag{15}$$

where erfc denotes the complementary error function, $\tilde{\mathcal{L}}^{(1)}$ can be written as

$$\begin{aligned}\tilde{\mathcal{L}}^{(1)}(B,a) = {} & \frac{m^3 b}{8\pi^{3/2} a}\int_0^\infty \frac{\mathrm{d}s}{s^{3/2}}\left(1-\frac{\lambda^2}{2s}\right)(\mathrm{e}^{-s}-\mathrm{erfc}(\sqrt{s}))\,\mathrm{e}^{-\lambda^2/4s}\coth(bs) \\ & + \frac{m^3 b}{\pi^2 a}\sum_{n=(0)}^{\infty}\int_0^\infty \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}(x\cos\lambda x) \\ & \times \log\left(1+\frac{\sqrt{x^2+2bn+1}-1}{\sqrt{x^2+2bn+1}+1}\,\mathrm{e}^{-2am\sqrt{x^2+2bn+1}}\right).\end{aligned} \tag{16}$$

Expressions (13) and (16) for the effective Lagrangian require the usual renormalization procedures. By expanding equation (13) in powers of B we find that all ultraviolet divergent terms are independent of B or proportional to B^2. The former can be simply subtracted out and the latter can be renormalized according to the usual procedure, namely, we define the renormalized charge e_r and the renormalized field B_r as $e_r = Z_3^{1/2}(\lambda)e$ and $B_r = Z_3^{-1/2}(\lambda)B$, where $Z_3^{1/2}(\lambda) = [1+\mathrm{e}^2(K_0(\lambda)-\lambda K_1(\lambda))/(6\pi^2)]^{-1/2}$ (K_ν denotes the modified Bessel function). We also find in (16) two terms which are subtracted out because they do not depend on B and as such do not contribute to the effective Lagrangian. The first is easily identified in the expansion of the function coth that appears in the first integral in (16). The second term may be identified by applying to the summation on n appearing in (16) the Euler–Maclaurin formula. The integral in this formula is obtained by converting the discrete index of summation n into a positive real variable of integration. By changing this variable of integration into the variable $\sqrt{x^2+2bn+1}$ the integral is brought to a form with manifest linear dependence on B, which is cancelled by the factor b in front of the summation on n appearing in (16). The remaining contributions from the Euler–Maclaurin formula provide the contributions to the effective Lagrangian in ascending powers of B. Actually, the contributions to the effective Lagrangian are in powers of B^2 due to the presence of the above-mentioned factor b multiplying the summation on n. The lowest order contribution will be used below in order to obtain the vacuum permeability. The ultraviolet finite terms in the effective Lagrangian are not affected by renormalization because they depend on the charge and magnetic field only through the product $eB = e_r B_r$. After the renormalization procedures the cut-off can be eliminated by taking the limit $\lambda \to 0$. The resulting properly renormalized expressions for $\mathcal{L}_{EH}^{(1)}$ and $\tilde{\mathcal{L}}^{(1)}$ are

$$\mathcal{L}_{EH}^{(1)}(B) = -\frac{m^4}{8\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\,\mathrm{e}^{-s}\left[bs\coth(bs)-1-\frac{(bs)^2}{3}\right] \tag{17}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}^{(1)}(B,a) = {} & \frac{m^3}{8\pi^{3/2}a}\int_0^\infty \frac{\mathrm{d}s}{s^{5/2}}[\mathrm{e}^{-s}-\mathrm{erfc}(\sqrt{s})][bs\coth(bs)-1] \\ & + \frac{m^3 b}{\pi^2 a}\sum_{n=(0)}^{\infty}\int_0^\infty \mathrm{d}x\,\log\left(1+\frac{\sqrt{x^2+2bn+1}-1}{\sqrt{x^2+2bn+1}+1}\,\mathrm{e}^{-2am\sqrt{x^2+2bn+1}}\right) \\ & - \frac{m^3}{\pi^2 a}\int_1^\infty \mathrm{d}x\, x\sqrt{x^2-1}\,\log\left(1+\frac{x-1}{x+1}\,\mathrm{e}^{-2amx}\right),\end{aligned} \tag{18}$$

where we have dropped the index r from the renormalized quantities.

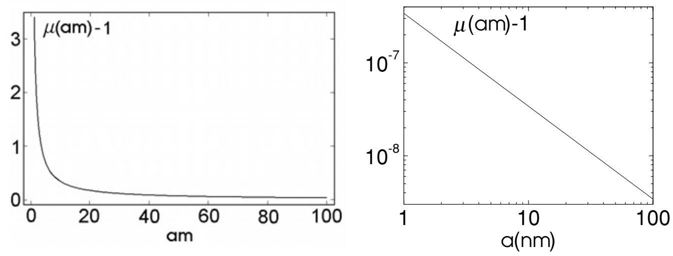

Figure 1. The figure at left shows the overall behaviour of permeability with distance in units of Compton wavelength and the figure at right shows the permeability for the electron field vacuum from one to one hundred of nanometres.

$\mathcal{L}^{(1)}_{EH}(B)$ is the usual Euler–Heisenberg effective Lagrangian density [1–3]. The lowest power in its Taylor expansion is B^4. This shows that the Dirac vacuum in the sole presence of an applied magnetic field exhibits nonlinear magnetic properties, but its permeability constant is $\mu = 1$, as in the case of the classical vacuum. The expression (18) for $\tilde{\mathcal{L}}^{(1)}(B, a)$ is our final result for the effective Lagrangian under MIT boundary conditions on parallel plates. The main feature of this effective Lagrangian is its dependence on the separation a of the plates. Now we can see that this confinement provides the Dirac vacuum with linear magnetic properties. Indeed, by expanding $\tilde{\mathcal{L}}^{(1)}(B, a)$ in powers of B we obtain from the B^2 term the following expression for the magnetic permeability constant of the vacuum:

$$\frac{1}{\mu(am)} = 1 - \frac{\pi - 2}{12\pi^2}\frac{e^2}{am} + \frac{1}{6\pi^2}\frac{e^2}{am}\int_1^\infty \mathrm{d}x \frac{x}{(x^2-1)^{3/2}} \log\left(1 + \frac{x-1}{x+1}\,\mathrm{e}^{-2amx}\right), \qquad (19)$$

which gives the permeability of the vacuum in the limit of small magnetic field. This expression is the main result of this paper. As in (1), we may also identify in the effective Lagrangian density $\tilde{\mathcal{L}}^{(1)}(B, a)$ the term $\tilde{\mathcal{L}}'^{(1)}(B, a)$ which is responsible for the nonlinear magnetic properties, but here we will concentrate on the magnetic permeability given by (19). It is a property that is probed by the presence of electromagnetic fields but depends solely on the separation between the confining plates. Let us remind that our result (18) for the effective action was obtained by the Weisskopf original method and it would be interesting to obtain it by other procedures as, e.g., the summation method of [11].

3. Discussion

The magnetic permeability of the vacuum as a function of the separation between the plates is plotted in figure 1. From expression (19) for the permeability it is obvious that the natural unit for the separation is the Compton wavelength $1/m$. On the range displayed in figure 1 we have $\mu(am) > 1$, which shows that the vacuum of the confined Dirac field behaves as a paramagnetic medium. In this respect we note that the vacuum becomes a diamagnetic medium if the plates' separation is smaller than $a_{cr} = (\pi\,\mathrm{e}^{-\gamma}/2m)\,\mathrm{e}^{-6\pi^2/e^2}$ (γ is the Euler constant). This phase transition was also observed in the case of anti-periodic boundary conditions [8], but since it occurs at the scale of the Landau pole we will not dwell further on its physical meaning. We also note that (for $a > a_{cr}$) the permeability decreases monotonically with the plates separation and it tends to 1 in the limit $a \to \infty$, which are physically sensible features.

For very small or very large separation between the plates the permeability (19) is given by the approximated expressions

$$\frac{1}{\mu(am)} = 1 - \frac{e^2}{6\pi^2}\log\left(\frac{\pi\,\mathrm{e}^{-\gamma}}{2am}\right) \quad (am \ll 1) \qquad \text{and}$$
$$\frac{1}{\mu(am)} = 1 - \frac{\pi-2}{12\pi^2}\frac{e^2}{am} \quad (am \gg 1). \tag{20}$$

To get a feeling for the order of magnitude of the permeability (19), we now make some numerical estimates. Let us first consider the vacuum of the electron field in a range of plates' separation going from nanometres to micrometres. In the case of anti-periodic boundary conditions, the changes in the magnetic permeability of the vacuum are beyond any possibility of observation, due to its exponential decay with am [8]. However, for the present case of confinement between plates the change $\Delta\mu \equiv \mu - 1$ in the permeability of the vacuum decays as $(am)^{-1}$. As a consequence we, obtain $\Delta\mu \sim 10^{-7}$ at a separation of one nanometre and $\Delta\mu \sim 10^{-9}$ at a separation of one-tenth of a micrometre (see figure 1). These values are comparable with those of hydrogen or nitrogen, which are of the order of 10^{-9} at room temperature and atmospheric pressure.

Now let us consider the range of plates separation at Fermi scales. If we take for a the value of 1 Fermi and for the charge and mass of the field the values for the u quark, the resulting change in permeability (19) is given by $\Delta\mu \sim 10^{-3}$. It is tempting to use our results, obtained for two plates separated by a distance a, to estimate the order of magnitude of the effect for quarks in a hadron, taking for a the radius of the latter. (Of course, such an estimate must not be taken too seriously—for instance, the Casimir pressure on the surface of a sphere of radius a has the same order of magnitude of the Casimir pressure on two parallel plates a distance a apart, but opposite sign.) From the estimate $\Delta\mu \sim 10^{-3}$ we find that the vacuum of the u quark confined in a hadron would exhibit a magnetic moment of the order of 10^{-14} nuclear magnetons in a magnetic field of 20 T. Even for such an extremely strong magnetic field the magnetic moment is still below the precision of present day experiments, of the order of $10^{-9}\mu_N$. It is a minute effect compared with the permanent magnetization of barions, but it could eventually have some significance for mesons. At any rate, such a magnetization is one more property to be added to the rich and complex structure of QCD vacuum.

4. Conclusion

In this paper we computed the effective Lagrangian for an applied magnetic field on the Dirac vacuum confined between parallel plates. The impermeability of the plates to the Dirac current is implemented by the MIT boundary conditions. The obtained effective Lagrangian gives rise to a magnetic permeability constant (19) for the Dirac vacuum, a feature which is absent in the original Euler–Heisenberg effective Lagrangian. The permeability constant depends on the separation between the parallel plates and has the expected overall behaviour, namely, the susceptibility falls off as the separation increases and goes to zero in the limit of infinite separation. Estimations of the order of magnitude of the susceptibility lead us to a value comparable with those of hydrogen or nitrogen at room temperature and atmospheric pressure for separations around 100 nanometres. It would be interesting to investigate the effect of such susceptibility on the speed of light. For a separation of one nanometre the contribution of the magnetic susceptibility would be of the order of 10^{-7}, instead of 10^{-36} in the Scharnhorst effect for the electromagnetic vacuum. Such a magnification could be explained in part by the fact that the usual Scharnhorst effect occurs at the two-loop level, while the one considered here already appears at the one-loop level. On the other hand, the renormalization of the distance between the plates obtained by Bordag [18] and the change (20) in vacuum permeability that we obtained for $am \gg 1$ are both of the same order of magnitude and proportional to the first power of the fine structure constant.

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